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Standard Tableaux and Modular Major Index

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Abstract. We provide simple necessary and sufficient conditions for the existence of a standard Young tableau of a given shape and major index r mod n, for all r. Our result generalizes the $r = 1$ case due essentially to Klyachko (1974) and proves a recent conjecture due to Sundaram (2016) for the $r = 0$ case. A byproduct of the proof is an asymptotic equidistribution result for "almost all" shapes. The proof uses a representation-theoretic formula involving Ramanujan sums and normalized symmetric group character estimates. Further estimates involving "opposite" hook lengths are given which are well-adapted to classifying which partitions $\lambda \vdash n$ have $f^\lambda \leq n^d$ for fixed d.

Keywords: standard Young tableaux, symmetric group characters, major index, hook length formula

1 Introduction

Let $\lambda \vdash n$ be an integer partition of size n, and let $\mathrm{SYT}(\lambda)$ denote the set of standard Young tableaux of shape λ. Let $\operatorname{maj} T$ denote the major index of $T \in \mathrm{SYT}(\lambda)$, namely the sum of all i for which $i+1$ appears below i (in English notation). We are chiefly interested in the counts

$$a_{\lambda,r} := \#\{T \in \mathrm{SYT}(\lambda) : \operatorname{maj} T \equiv_n r\}$$

where r is taken mod n. To avoid giving undue weight to trivial cases, we take $n \geq 1$ throughout. Work due to Klyachko and, later, Kraskiewicz-Weyman, gives the following:

Theorem 1.1 ([5, Proposition 2], [6]). *Let $\lambda \vdash n$ and $n \geq 1$. Then $a_{\lambda,1}$ is positive except in the following cases, when it is zero:*

- $\lambda = (2,2)$ *or* $\lambda = (2,2,2)$*;*
- $\lambda = (n)$ *when* $n > 1$*; or* $\lambda = (1^n)$ *when* $n > 2$.

Indeed, the $a_{\lambda,r}$ have a natural interpretation as irreducible multiplicities as follows, a result originally due to Kraskiewicz-Weyman. Let C_n be the cyclic group of order n generated by the long cycle $\sigma_n := (12 \cdots n) \in S_n$, let S^λ be the Specht module of shape

$\lambda \vdash n$, and let $\chi^r \colon C_n \to \mathbb{C}^\times$ be the irreducible representation given by $\chi^r(\sigma_n^i) := \omega_n^{ri}$ where ω_n is a fixed primitive nth root of unity and $r \in \mathbb{Z}/n$. Let $\langle -, - \rangle$ denote the standard scalar product for complex representations.

Theorem 1.2 (see [6, Theorem 1]). *With the above notation, we have*

$$\langle S^\lambda, \chi^r\uparrow_{C_n}^{S_n} \rangle = a_{\lambda,r} = \langle \chi^r, S^\lambda\downarrow_{C_n}^{S_n} \rangle.$$

Moreover, $a_{\lambda,r}$ depends only on λ and $\gcd(n,r)$, i.e. if $\gcd(n,r) = \gcd(n,s)$ then $a_{\lambda,r} = a_{\lambda,s}$.

Remark 1.3. Kraskiewicz-Weyman gave the first equality in Theorem 1.2, and the second follows by Frobenius reciprocity. Klyachko [5, Proposition 2] actually determined which S^λ contain faithful representations of C_n in agreement with Theorem 1.1. One may see through a variety of methods that $\chi^r\uparrow_{C_n}^{S_n}$ depends up to isomorphism only on $\gcd(r,n)$.

The manuscript [6] was long-unpublished, the delay being largely due to Klyachko having already given a significantly more direct proof of their main application, relating $\chi^1\uparrow_{C_n}^{S_n}$ to free Lie algebras, though we have no need of this connection. For a more modern and unified account of these results, see [8, Theorems 8.8-8.12].

The following conjecture due to Sundaram was originally stated in terms of the multiplicity of S^λ in $1\uparrow_{C_n}^{S_n}$.

Conjecture 1.4 ([13]). *Let $\lambda \vdash n$ and $n \geq 1$. Then $a_{\lambda,0}$ is positive except in the following cases, when it is zero: $n > 1$ and*

- $\lambda = (n-1, 1)$
- $\lambda = (2, 1^{n-2})$ *when n is odd*
- $\lambda = (1^n)$ *when n is even.*

Conjecture 1.4 is the $r = 0$ case of the following theorem, which is our main result.

Theorem 1.5. *Let $\lambda \vdash n$ and $n \geq 1$. Then $a_{\lambda,r}$ is positive except in the following cases, when it is zero: $n > 1$ and*

- $\lambda = (2,2)$, $r = 1,3$; *or* $\lambda = (2,2,2)$, $r = 1,5$; *or* $\lambda = (3,3)$, $r = 2,4$;
- $\lambda = (n-1,1)$ *and* $r = 0$;
- $\lambda = (2, 1^{n-2})$, $r = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ is even;} \end{cases}$
- $\lambda = (n)$, $r \in \{1, \ldots, n-1\}$;

- $\lambda = (1^n)$, $r \in \begin{cases} \{1, \ldots, n-1\} & \text{if } n \text{ is odd} \\ \{0, \ldots, n-1\} - \{\frac{n}{2}\} & \text{if } n \text{ is even.} \end{cases}$

Equivalently, using Theorem 1.2, every irreducible representation appears in each $\chi^r\uparrow_{C_n}^{S_n}$ or $S^\lambda\downarrow_{C_n}^{S_n}$ except in the noted exceptional cases.

Our main tool is the following well-known representation-theoretic formula. See Section 2 for further discussion of its origins and a generalization. Let $\chi^\lambda(\mu)$ denote the character of S^λ at a permutation of cycle type μ. Write $f^\lambda := \chi^\lambda(1^n) = \dim S^\lambda = \#\mathrm{SYT}(\lambda)$.

Theorem 1.6. *Let $\lambda \vdash n$ and $n \geq 1$. For all $r \in \mathbb{Z}/n$,*

$$\frac{a_{\lambda,r}}{f^\lambda} = \frac{1}{n} + \frac{1}{n} \sum_{\substack{\ell | n \\ \ell \neq 1}} \frac{\chi^\lambda(\ell^{n/\ell})}{f^\lambda} c_\ell(r)$$

where

$$c_\ell(r) := \mu\left(\frac{\ell}{\gcd(\ell, r)}\right) \frac{\phi(\ell)}{\phi(\ell / \gcd(\ell, r))}$$

is a Ramanujan sum, μ is the classical Möbius function, and ϕ is Euler's totient function.

We estimate the quotients in the preceding formula using the following result due to Fomin and Lulov. A *ribbon* is a connected skew shape with no 2×2 rectangles.

Theorem 1.7 ([2, Theorem 1.1])**.** *Let $\lambda \vdash n$ where $n = \ell s$. Suppose λ can be written as s successive ribbons each of length ℓ. Then*

$$|\chi^\lambda(\ell^s)| \leq \frac{s!\ell^s}{(n!)^{1/\ell}} (f^\lambda)^{1/\ell}.$$

Theorem 1.7 is based on the following generalization of the hook length formula (the $\ell = 1$ case), which seems less well-known than it deserves. For $\lambda \vdash n$, write $c \in \lambda$ to mean that c is a cell in λ. Further write h_c for the *hook length* of c and write $[n] := \{1, 2, \ldots, n\}$.

Theorem 1.8 ([2, Corollary 2.2]; see also [4, p. 2.7.32])**.** *Let $\lambda \vdash n$ where $n = \ell s$. Then*

$$|\chi^\lambda(\ell^s)| = \frac{\prod_{\substack{i \in [n] \\ i \equiv_\ell 0}} i}{\prod_{\substack{c \in \lambda \\ h_c \equiv_\ell 0}} h_c} \tag{1.1}$$

whenever λ can be written as s successive ribbons of length ℓ, and 0 otherwise.

We also give the following asymptotic uniform distribution result which largely strengthens Theorem 1.5.

Theorem 1.9. *Let $\lambda \vdash n$ be a partition where $f^\lambda \geq n^5 \geq 1$. Then for all r,*

$$\left| \frac{a_{\lambda,r}}{f^\lambda} - \frac{1}{n} \right| < \frac{1}{n^2}.$$

In particular, if $n \geq 81$, $\lambda_1 < n-7$, and $\lambda'_1 < n-7$, then $f^\lambda \geq n^5$ and the inequality holds.

Indeed, the upper bound in Theorem 1.9 is quite weak and is intended only to convey the flavor of the distribution of $(a_{\lambda,r})_{r=0}^{n-1}$ for fixed λ. One may use Roichman's asymptotic estimate [9] of $|\chi^\lambda(\ell^s)|/f^\lambda$ to prove exponential decay in many cases. Moreover, one typically expects f^λ to grow super-exponentially, i.e. like $(n!)^\epsilon$ for some $\epsilon > 0$ (see [7] for some discussion and a more recent generalization of Roichman's result), which in turn would give a super-exponential decay rate in Theorem 1.9. We have no need for such refined statements and so have not pursued them further.

The rest of the paper is organized as follows. In Section 2 we discuss and generalize Theorem 1.6. In Section 3, we use symmetric group character estimates and a new estimate involving "opposite hook products," Lemma 3.4, to deduce our main results, Theorem 1.5 and Theorem 1.9. We have omitted proofs from this extended abstract. They will appear in a forthcoming version of this article [14].

2 Generalizing the Main Formula

Variations on Theorem 1.6 have appeared in the literature numerous times in several guises, sometimes implicitly (see [1, Théorème 2.2], [5, (7)], or [12, 7.88(a), p. 541]). In this section we write out a precise and relatively general version of these results which explicitly connects Theorem 1.6 to the well-known corresponding symmetric function expansion due to H. O. Foulkes. Let ch denote the Frobenius characteristic map, and let p_λ denote the power symmetric function indexed by the partition λ.

Theorem 2.1 ([3, Theorem 1])**.** *Suppose $\lambda \vdash n \geq 1$ and $r \in \mathbb{Z}/n$. In this case,*

$$\operatorname{ch} \chi^r \uparrow_{C_n}^{S_n} = \frac{1}{n} \sum_{\ell | n} c_\ell(r) p_{(\ell^{n/\ell})}. \tag{2.1}$$

The following straightforward result connects and generalizes Theorem 2.1 and Theorem 1.6.

Theorem 2.2. *Let H be a subgroup of S_n, and let M be a finite-dimensional H-module with character $\chi^M \colon H \to \mathbb{C}$. Then*

$$\mathrm{ch}\, M\!\uparrow_H^{S_n} = \frac{1}{|H|} \sum_{\mu \vdash n} c_\mu p_\mu \tag{2.2}$$

and, for all $\lambda \vdash n$,

$$\langle M\!\uparrow_H^{S_n}, S^\lambda \rangle = \frac{1}{|H|} \sum_{\mu \vdash n} c_\mu \chi^\lambda(\mu), \tag{2.3}$$

where

$$c_\mu := \sum_{\substack{h \in H \\ \tau(h) = \mu}} \chi^M(h)$$

and $\tau(\sigma)$ denotes the cycle type of the permutation σ.

Theorem 2.2 is an immediate consequence of the induced character formula. Note that (2.2) specializes to Theorem 2.1 and (2.3) specializes to Theorem 1.6 when $M = \chi^r$. In that case, the only possibly non-zero c_μ arise from $\mu = (\ell^{n/\ell})$ for $\ell \mid n$.

One may consider analogues of the counts $a_{\lambda,r}$ obtained by inducing other one-dimensional representations of subgroups of S_n. Motivated by the study of so-called higher Lie modules, there is a natural embedding of reflection groups $C_a \wr S_b \hookrightarrow S_{ab}$. A classification analogous to Klyachko's result, Theorem 1.1, was asserted for $b = 2$ by Schocker [10, Theorem 3.4], though the "rather lengthy proof" making "extensive use of routine applications of the Littlewood-Richardson rule and some well-known results from the theory of plethysms" was omitted. By contrast, our approach using Theorem 2.2 may be pushed through in this case using an appropriate generalization of the Fomin-Lulov bound, Theorem 1.7, such as [7, Theorem 1.1], resulting in analogues of Theorem 1.5 and Theorem 1.9. Our approach begins to break down when b is large relative to $n = ab$ and (2.3) has many terms. However, we have no current need for such generalizations and so have not pursued them further.

3 Proof of the Main Result

We now summarize our proof of Theorem 1.5 and Theorem 1.9. We begin by giving a sufficient condition in terms of upper bounds on symmetric group character ratios, Lemma 3.1, which in turn reduces to a sufficient condition in terms of lower bounds on f^λ, Corollary 3.2. We then give an inequality between hook length products and "opposite" hook length products, Lemma 3.4, from which one can classify λ for which $f^\lambda < n^3$. Theorem 1.5 follows in almost all cases, with the remainder being handled by brute force and case-by-case analysis. Theorem 1.9 is similar, except the bound $f^\lambda < n^5$ is used.

Lemma 3.1. *Let $\lambda \vdash n$ and $d \in \mathbb{R}$. Suppose for all $1 \neq \ell \mid n$ where λ may be written as $s := n/\ell$ successive ribbons each of length ℓ that*

$$\frac{|\chi^\lambda(\ell^s)|}{f^\lambda} \leq \frac{1}{n^d \phi(\ell)}. \tag{3.1}$$

Then for all $r \in \mathbb{Z}/n$,

$$\left| \frac{a_{\lambda,r}}{f^\lambda} - \frac{1}{n} \right| < \frac{1}{n^d}.$$

Lemma 3.1 follows from Theorem 1.6. The following corollary to Lemma 3.1 follows from Theorem 1.7 and Stirling's approximation [11, (1.53)].

Corollary 3.2. *Let $\lambda \vdash n$. If $f^\lambda \geq n^3 \geq 1$, then $a_{\lambda,r} \neq 0$.*

We next summarize techniques that are well-adapted to classifying $\lambda \vdash n$ for which $f^\lambda < n^d$ for fixed d. We begin with a curious observation, Lemma 3.4, which we have not been able to locate in the literature (though contrast it with [2, Theorem 2.3]).

Definition 3.3. Consider a partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ as a set of cells

$$\lambda = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : 1 \leq b \leq m, 1 \leq a \leq \lambda_b\}.$$

Given a cell $c = (a, b) \in \lambda \subset \mathbb{N} \times \mathbb{N}$, the *opposite hook length* h_c^{op} at c is $a + b - 1$. For instance, the unique cell in $\lambda = (1)$ has opposite hook length 1, and the opposite hook length increases by 1 for each north or east step (using French notation).

It is easy to see that $\sum_{c \in \lambda} h_c^{\mathrm{op}} = \sum_{c \in \lambda} h_c$. On the other hand, we have the following.

Lemma 3.4. *For all partitions λ,*

$$\prod_{c \in \lambda} h_c^{\mathrm{op}} \geq \prod_{c \in \lambda} h_c.$$

Moreover, equality holds if and only if λ is a rectangle.

Our proof of Lemma 3.4 involves considering contributions of the (co-)arm and (co-)leg lengths of each cell. It would be interesting to find a more conceptual explanation for Lemma 3.4, perhaps using representation theory. The appearance of rectangles is particularly striking. Note, however, that $n!/\prod_{c \in \lambda} h_c^{\mathrm{op}}$ need not be an integer. In any case, we continue towards Theorem 1.5.

Definition 3.5. Define the *diagonal preorder* on partitions as follows. Declare $\lambda \lesssim^{\mathrm{diag}} \mu$ if and only if for all $i \in \mathbb{P}$,

$$\#\{c \in \lambda : h_c^{\mathrm{op}} \geq i\} \leq \#\{d \in \mu : h_d^{\mathrm{op}} \geq i\}.$$

Note that $\lesssim^{\mathrm{diag}}$ is reflexive and transitive, though not anti-symmetric, so the diagonal preorder is not a partial order. A straightforward consequence of the definition is that

$$\lambda \lesssim^{\mathrm{diag}} \mu \qquad \Rightarrow \qquad \prod_{c \in \lambda} h_c^{\mathrm{op}} \leq \prod_{d \in \mu} h_d^{\mathrm{op}}. \tag{3.2}$$

Hooks are maximal elements of the diagonal preorder in a sense we next make precise.

Definition 3.6. Let $\lambda \vdash n$ for $n \geq 1$. The *diagonal excess* of λ is

$$N(\lambda) := |\lambda| - \#\{h_c^{\mathrm{op}} : c \in \lambda\}.$$

For instance, $\lambda = (3,3)$ has opposite hook lengths ranging from 1 to 4, so $N((3,3)) = 6 - 4 = 2$.

Example 3.7. Let $\lambda \vdash n$ be a hook. Consider the sequence $(\#\{c \in \lambda : h_c^{\mathrm{op}} = i\})_{i=1}^{\infty}$ recording the number of cells with opposite hook lengths $1, 2, 3, \ldots$. This sequence is

$$(1, 2, 2, \ldots, 2, 1, \ldots, 1, 0, 0, \ldots)$$

where there are $N(\lambda)$ two's and $n - N(\lambda)$ non-zero entries. In particular, $N(\lambda) + 1 \leq n - N(\lambda)$, i.e. $2N(\lambda) + 1 \leq n$.

Proposition 3.8. *Let $\lambda \vdash n$ for $n \geq 1$. Set*

$$N := \begin{cases} N(\lambda) & \text{if } 2N(\lambda) + 1 \leq n \\ \left\lfloor \frac{n-1}{2} \right\rfloor & \text{if } 2N(\lambda) + 1 > n. \end{cases} \tag{3.3}$$

Then

$$\lambda \lesssim^{\mathrm{diag}} (n - N, 1^N). \tag{3.4}$$

In particular, if $2N(\lambda) + 1 \leq n$, then the hook $(n - N(\lambda), 1^{N(\lambda)})$ is maximal for the diagonal preorder on partitions of size n with diagonal excess $N(\lambda)$.

Our proof of Proposition 3.8 is algorithmic. Each step of the algorithm goes up in the diagonal preorder and the algorithm terminates at an appropriate hook. The following corollary of Proposition 3.8 and Lemma 3.4 essentially gives a polynomial lower bound on f^λ in terms of the diagonal excess.

Corollary 3.9. *Let $\lambda \vdash n$ for $n \geq 1$, and take N as in (3.3). For any $0 \leq M \leq N$, we have*

$$\prod_{c \in \lambda} h_c^{\mathrm{op}} \leq (n - M)!(M + 1)!. \tag{3.5}$$

Indeed,

$$f^\lambda \geq \frac{1}{M+1}\binom{n}{M}. \tag{3.6}$$

We now sketch the proof of Theorem 1.5, the proof of Theorem 1.9 being similar. Theorem 1.5 follows from Corollary 3.2 except when $f^\lambda < n^3$. One may classify these exceptional λ using the bound (3.6) from Corollary 3.9 for n sufficiently large as essentially those λ with $N(\lambda) \leq 4$. The result is twelve pairs of infinite families, namely the concatenations $(n-M) \oplus \mu$ for $\mu \vdash M \leq 4$ and their conjugates. For example, one such pair is $\{(n-4,3,1)\}$ and its conjugate. The five pairs with $M = 4$ all result in $f^\lambda \geq n^3$ for $n \geq 34$. The conclusion of Theorem 1.5 may be verified by hand for the remaining seven families for $n \geq 15$. One must then verify the conclusion of Theorem 1.5 for $n \leq 33$, which takes little time on modern computers.

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References

[1] J. Désarménien. “Étude modulo n des statistiques mahoniennes”. *Sém. Loth. Combin.* **22** (1990), Art. B22a. URL.

[2] S. Fomin and N. Lulov. “On the number of rim hook tableaux”. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* **223** (1995), pp. 219–226. DOI.

[3] H. O. Foulkes. “Characters of symmetric groups induced by characters of cyclic subgroups”. *Combinatorics (Proc. Conf. Combinatorial Math., Math. Inst., Oxford, 1972)*. Inst. Math. Appl., Southend-on-Sea, 1972, pp. 141–154.

[4] G. James and A. Kerber. *The Representation Theory of the Symmetric Group*. Encyclopedia of Mathematics and its Applications, Vol. 16. Addison-Wesley, 1981.

[5] A. A. Klyachko. “Lie elements in the tensor algebra”. *Siberian Math. J.* **15** (1974), pp. 914–920. DOI.

[6] W. Kraśkiewicz and J. Weyman. “Algebra of coinvariants and the action of a Coxeter element”. *Bayreuth. Math. Schr.* **63** (2001), pp. 265–284.

[7] M. Larsen and A. Shalev. “Characters of symmetric groups: sharp bounds and applications”. *Invent. Math.* **174** (2008), pp. 645–687. DOI.

[8] C. Reutenauer. *Free Lie Algebras*. London Math. Soc. Monographs. New Series, Vol. 7. The Clarendon Press, Oxford University Press, 1993.

[9] Y. Roichman. “Upper bound on the characters of the symmetric groups”. *Invent. Math.* **125** (1996), pp. 451–485. DOI.

[10] M. Schocker. “Embeddings of higher Lie modules”. *J. Pure Appl. Algebra* **185** (2003), pp. 279–288. DOI.

[11] J. Spencer. *Asymptopia*. Student Mathematical Library, Vol. 71. Amer. Math. Soc., 2014. DOI.

[12] R. P. Stanley. *Enumerative Combinatorics, Vol. 2*. Cambridge Studies in Advanced Mathematics, Vol. 62. Cambridge University Press, 1999. DOI.

[13] S. Sundaram. “On Conjugacy Classes of S_n Containing all Irreducibles”. Preprint. 2016.

[14] J. P. Swanson. “On the Existence of Tableaux with Given Modular Major Index”. In preparation. 2016.